Generalized Toric Varieties for Simple Non-Rational Convex Polytopes

Fiammetta Battaglia and Elisa Prato

Abstract

We call *complex quasifold* of dimension k a space that is locally isomorphic to the quotient of an open subset of the space $\mathbb{C}^k$ by the holomorphic action of a discrete group; the analogue of a complex torus in this setting is called a *complex quasitorus*. We associate to each simple polytope, rational or not, a family of complex quasifolds having same dimension as the polytope, each containing a dense open orbit for the action of a suitable complex quasitorus. We show that each of these spaces M is diffeomorphic to one of the symplectic quasifolds defined in [P], and that the induced symplectic structure is compatible with the complex one, thus defining on M the structure of a Kähler quasifold. These spaces may be viewed as a generalization of the toric varieties that are usually associated to those simple convex polytopes that are rational.

Introduction

Consider a vector space $\mathfrak{d}$ of dimension n. To each simple convex polytope $\Delta \subset \mathfrak{d}^*$ that is rational with respect to a lattice in $\mathfrak{d}$ there corresponds a toric variety with at worst quotient singularities. What happens in the case that the simple convex polytope is no longer rational? To answer this question we consider a special class of spaces, called *quasifolds*, which were first introduced by one of the authors in [P]. A quasifold is not necessarily a Hausdorff space: it is locally modeled by orbit spaces of the action of discrete, possibly infinite, groups on open subsets of $\mathbb{R}^k$. A *quasitorus*, on the other hand, is the natural replacement of a torus in this geometry.

In this article we define the notions of complex quasifold and complex quasitorus and we associate to each simple convex polytope $\Delta \subset \mathfrak{d}^*$ a family of compact complex quasifolds of dimension n, each endowed with the holomorphic action of a complex quasitorus $D_{\mathbb{C}}$ having a dense open orbit. Our construction is explicit: each space M is the topological quotient of a suitable open subset of $\mathbb{C}^d$ by the action of a suitable subgroup $N_{\mathbb{C}} \subset T^d_{\mathbb{C}} = \mathbb{C}^d/\mathbb{Z}^d$, and $D_{\mathbb{C}}$ is isomorphic to $T^d_{\mathbb{C}}/N_{\mathbb{C}}$, d being the number of facets of the polytope. We show that M is a complex quasifold by covering it with mutually compatible local models of the type $\mathbb{C}^n$ modulo the action of a discrete group. If the polytope is rational our procedure matches the standard one for constructing toric varieties as quotients that is described in Appendix 1 of Guillemin's book [G].

It is shown in [P] that to the same simple convex polytope Δ one can also associate a family of symplectic quasifolds of dimension $2n$, each endowed with the effective Hamiltonian action of a quasitorus D, having the property that the image of the corresponding moment mapping is exactly Δ. The construction extends the one given by Delzant [D] in the smooth case: it is explicit and uses the symplectic quotient operation. In the last section of this paper we show that each complex quotient M is equivariantly diffeomorphic to one of these symplectic quotients, that the induced symplectic structure is compatible with the complex one, and thus defines on M the structure of a Kähler quasifold; of course $D_{\mathbb{C}}$ here is the complexification of the corresponding quasitorus D.

For these reasons these spaces may well be thought of as a natural generalization of toric varieties for simple convex polytopes that are not rational.

1   Complex quasifolds and complex quasitori

This section is devoted to defining complex quasifolds and their geometry. We will not repeat the remarks and results that are in common with the real case, for which we refer the reader to the article [P]. The local model for complex quasifolds is a complex manifold acted on holomorphically by a discrete group.

Definition 1.1 (Complex model) Let $\tilde{V}$ be a connected, simply connected complex manifold of dimension k and let Γ be a discrete group acting on $\tilde{V}$ holomorphically so that the set of points, $\tilde{V}_0$, where the action is free, is connected and dense. Consider the space of orbits, $\tilde{V}/\Gamma$, of the action of the group Γ on the manifold $\tilde{V}$, endowed with the quotient topology, and the canonical projection $p: \tilde{V} \to \tilde{V}/\Gamma$. A *complex model* of dimension k is the triple $(\tilde{V}/\Gamma, p, \tilde{V})$, shortly $\tilde{V}/\Gamma$.

Remark 1.2 We remark that the assumption in Definition 1.1 that the manifold $\tilde{V}$ be simply connected is not as strong as one may think. Consider the triple $(\tilde{V}/\Gamma, p, \tilde{V})$ as defined above but assume that the manifold $\tilde{V}$ is not simply connected; consider its universal cover, $\pi: V^{\#} \to \tilde{V}$, and its fundamental group, Π. The manifold $V^{\#}$ is connected and simply connected, the mapping π is holomorphic, the discrete group Π acts holomorphically, freely and properly on the manifold $V^{\#}$ and $\tilde{V} = V^{\#}/\Pi$. Consider the extension of the group Γ by the group Π, $1 \longrightarrow \Pi \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow 1$, defined as follows

$$\Lambda = \left\{ \lambda \in \mathrm{Diff}(V^{\#}) \mid \exists\, \gamma \in \Gamma \text{ s. t. } \pi(\lambda(u^{\#})) = \gamma \cdot \pi(u^{\#}) \,\forall\, u^{\#} \in V^{\#} \right\}.$$

It is easy to verify that Λ is a discrete group, that it acts on the manifold $V^{\#}$ according to the assumptions of Definition 1.1 and that $\tilde{V}/\Gamma = V^{\#}/\Lambda$.

Definition 1.3 (Submodel) Consider a model $(\tilde{V}/\Gamma, p, \tilde{V})$ and let W be an open subset of $\tilde{V}/\Gamma$. We will say that W is a submodel of $(\tilde{V}/\Gamma, p, \tilde{V})$, if $(p^{-1}(W), p, W)$ defines a model according to Remark 1.2.

Definition 1.4 (Holomorphic mapping, biholomorphism of models) Given two models $(\tilde{V}/\Gamma, p, \tilde{V})$ and $(\tilde{W}/\Delta, q, \tilde{W})$, a mapping $f : \tilde{V}/\Gamma \longrightarrow \tilde{W}/\Delta$ is said to be *holomorphic* if there exists a holomorphic mapping $\tilde{f} : \tilde{V} \longrightarrow \tilde{W}$ such that $q \circ \tilde{f} = f \circ p$; we will say that $\tilde{f}$ is a *lift* of f. We will say that the holomorphic mapping f is a *biholomorphism of models* if it is bijective and if the lift $\tilde{f}$ is a biholomorphism.

If the mapping $\tilde{f}$ is a lift of a holomorphic mapping of models $f : \tilde{U}/\Gamma \longrightarrow \tilde{V}/\Delta$ so are the mappings $\tilde{f}^{\gamma}(-) = \tilde{f}(\gamma \cdot -)$, for all elements γ in Γ and ${}^{\delta}\tilde{f}(-) = \delta \cdot \tilde{f}(-)$, for all elements δ in Δ. If the mapping f is a biholomorphism, then these are the only other possible lifts and the groups Γ and Δ are isomorphic; the proof goes exactly as in the real case, see [P, orange and green lemmas].

Geometric objects on a model $\tilde{V}/\Gamma$ are defined by Γ-invariant geometric objects on the manifold $\tilde{V}$. For example:

Definition 1.5 (Differential form on a model) A *differential form of degree h*, ω, on a model $\tilde{V}/\Gamma$ is the assignment of a Γ-invariant differential form of degree h, $\tilde{\omega}$, on the complex manifold $\tilde{V}$.

Definition 1.6 (Kähler form on a model) A *Kähler form* on a model $\tilde{V}/\Gamma$ is a differential form, ω, such that $\tilde{\omega}$ (see Definition 1.5) is Kähler.

Complex quasifolds are obtained by gluing together the models in the appropriate way:

Definition 1.7 (Complex quasifold) A dimension k *complex quasifold structure* on a topological space M is the assignment of an *atlas*, or collection of *charts*, $\mathcal{A} = \{ (V_{\alpha}, \phi_{\alpha}, \tilde{V}_{\alpha}/\Gamma_{\alpha}) \,|\, \alpha \in A \}$ having the following properties:

1. The collection $\{ V_{\alpha} \,|\, \alpha \in A \}$ is a cover of M.

2. For each index α in A, the set V_{α} is open, the space $\tilde{V}_{\alpha}/\Gamma_{\alpha}$ defines a model, where the set $\tilde{V}_{\alpha}$ is an open, connected, and simply connected subset of the space $\mathbb{C}^{k}$, and the mapping ϕ_{α} is a homeomorphism of the space $\tilde{V}_{\alpha}/\Gamma_{\alpha}$ onto the set V_{α}.

3. For all indices α, β in A such that $V_{\alpha} \cap V_{\beta} \neq \emptyset$, the sets $\phi_{\alpha}^{-1}(V_{\alpha} \cap V_{\beta})$ and $\phi_{\beta}^{-1}(V_{\alpha} \cap V_{\beta})$ are submodels of $\tilde{V}_{\alpha}/\Gamma_{\alpha}$ and $\tilde{V}_{\beta}/\Gamma_{\beta}$ respectively and the mapping

$$g_{\alpha\beta} = \phi_{\beta}^{-1} \circ \phi_{\alpha} : \phi_{\alpha}^{-1}(V_{\alpha} \cap V_{\beta}) \longrightarrow \phi_{\beta}^{-1}(V_{\alpha} \cap V_{\beta})$$

   is a biholomorphism of models. We will then say that the mapping $g_{\alpha\beta}$ is a *change of charts* and that the corresponding charts are *compatible*.

4. The atlas $\mathcal{A}$ is maximal, that is: if the triple $(V, \phi, \tilde{V}/\Gamma)$ satisfies property 2. and is compatible with all the charts in $\mathcal{A}$, then $(V, \phi, \tilde{V}/\Gamma)$ belongs to $\mathcal{A}$.

We will say that a space M with a complex quasifold structure is a *complex quasifold*.

A complex quasifold of dimension k has an underlying structure of real quasifold of dimension $2k$.

Remark 1.8 To each point $m \in M$ there corresponds a discrete group Γ_m defined as follows: take a chart $(V_\alpha, \phi_\alpha, \tilde{V}_\alpha/\Gamma_\alpha)$ around m, then Γ_m is the isotropy group of Γ_α at any point $\tilde{v} \in \tilde{V}$ which projects down to m. One can check that this definition does not depend on the choice of the chart.

Definition 1.9 (Holomorphic mapping, biholomorphism) Let M and N be two complex quasifolds. A continuous mapping $f \colon M \longrightarrow N$ is said to be a *holomorphic mapping* if there exists a chart $(V_\alpha, \phi_\alpha, \tilde{V}_\alpha/\Gamma_\alpha)$ around each point m in the space M, a chart $(W_\alpha, \psi_\alpha, \tilde{W}_\alpha/\Delta_\alpha)$ around the point $f(m)$, and a holomorphic mapping of models $f_\alpha \colon \tilde{V}_\alpha/\Gamma_\alpha \to \tilde{W}_\alpha/\Delta_\alpha$ such that $\psi_\alpha \circ f_\alpha = f \circ \phi_\alpha$. If f is bijective, and if each f_α is a biholomorphism of models, we will say that f is a *biholomorphism*.

We remark that two biholomorphic quasifolds are also diffeomorphic with respect to their underlying real quasifold structure.

Geometric objects on quasifolds are defined as geometric objects on the charts that respect the changes of charts. For example:

Definition 1.10 (Differential form) A *differential form of degree h*, ω, on a complex quasifold M is the assignment of a chart $(V_\alpha, \phi_\alpha, \tilde{V}_\alpha/\Gamma_\alpha)$ around each point $m \in M$ and of a differential form of degree h, ω_α, on the model $\tilde{V}_\alpha/\Gamma_\alpha$. We require that whenever we have two such charts, $(V_\alpha, \phi_\alpha, \tilde{V}_\alpha/\Gamma_\alpha)$ and $(V_\beta, \phi_\beta, \tilde{V}_\beta/\Gamma_\beta)$, with the property that $V_\alpha \cap V_\beta \neq \emptyset$, then $(g_{\alpha\beta})^* \omega_\beta = \omega_\alpha$ for the corresponding change of charts $g_{\alpha\beta}$.

Definition 1.11 (Kähler form) A *Kähler form*, ω, on a complex quasifold M is a differential form, ω, such that each ω_α (see Definition 1.10) is a Kähler form on the model $\tilde{V}_\alpha/\Gamma_\alpha$. A *Kähler structure* on a quasifold M is the assignment of a Kähler form, ω, and we will say that (M, ω), or simply M, is a *Kähler quasifold*.

The analogue of a torus in the real setting is a quasitorus. Let $\mathfrak{d}$ be a vector space of dimension n. We recall from [P] that a quasitorus of dimension n and quasi-Lie algebra $\mathfrak{d}$ is the quotient of the space $\mathfrak{d}$ by a quasilattice Q, which in turn is the $\mathbb{Z}$-span of a set of spanning vectors in $\mathfrak{d}$. Consider now the complexification of $\mathfrak{d}$, the complex vector space $\mathfrak{d}_\mathbb{C} = \mathfrak{d} + i\mathfrak{d}$; the quasilattice Q acts naturally on $\mathfrak{d}_\mathbb{C}$:

$$\begin{array}{ccccc} Q & \times & \mathfrak{d}_\mathbb{C} & \longrightarrow & \mathfrak{d}_\mathbb{C} \\ (A & , & X + iY) & \longmapsto & (X + A) + iY. \end{array} \tag{1}$$

Definition 1.12 (Complex quasitorus, quasi-Lie algebra, exponential) Let $\mathfrak{d}$ be a vector space of dimension n and let Q be a quasilattice in $\mathfrak{d}$. Then we call the quotient $D_\mathbb{C} = \mathfrak{d}_\mathbb{C}/Q$ *complex quasitorus* of dimension n and *quasi-Lie algebra* $\mathfrak{d}_\mathbb{C}$. We call the corresponding projection $\mathfrak{d}_\mathbb{C} \to D_\mathbb{C}$ *exponential mapping* and we denote it by exp.

If the quasilattice Q is a lattice L we obtain the honest complex torus $\mathfrak{d}_\mathbb{C}/L$, which is the complexification of the torus $\mathfrak{d}/L$. The complex quasitorus $D_\mathbb{C}$ is a quasifold of one chart and may be naturally thought of as the complexification of the quasitorus $D = \mathfrak{d}/Q$. The main result that we will be needing is the following:

Proposition 1.13 *Let T be a torus and N a Lie subgroup. Then $T_{\mathbb{C}}/N_{\mathbb{C}}$ is a complex quasitorus of complex dimension $n = \dim T - \dim N$.*

Proof. This proposition has a real analogue, the proof of which is quite similar (compare with [P, Proposition 2.5]). Denote by $\mathfrak{n}$ the Lie algebra of N, and by $\mathfrak{t}$ the Lie algebra of T. Let $\mathfrak{d}$ be a complement of $\mathfrak{n}$ in $\mathfrak{t}$, then the complex vector space $\mathfrak{d}_{\mathbb{C}} = \mathfrak{d} + i\mathfrak{d}$ is a complement of $\mathfrak{n}_{\mathbb{C}}$ in $\mathfrak{t}_{\mathbb{C}}$. We define a projection $p\colon \mathfrak{d}_{\mathbb{C}} \longrightarrow T_{\mathbb{C}}/N_{\mathbb{C}}$ by the rule $p(Z) = \Pi(\exp Z), Z \in \mathfrak{d}_{\mathbb{C}}$, where $\Pi\colon T_{\mathbb{C}} \longrightarrow T_{\mathbb{C}}/N_{\mathbb{C}}$ denotes the canonical projection. Notice that, by definition of exp the kernel of p is a quasilattice Q and p induces a group isomorphism $\mathfrak{d}_{\mathbb{C}}/Q \cong T_{\mathbb{C}}/N_{\mathbb{C}}$. $\qquad\square$

We conclude this section with the definition of holomorphic action of a complex quasitorus on a complex quasifold.

Definition 1.14 (Holomorphic action) A *holomorphic action* of a complex quasitorus $D_{\mathbb{C}}$ on a complex quasifold M is a holomorphic mapping $\tau\colon D_{\mathbb{C}} \times M \longrightarrow M$ such that $\tau(d_1 \cdot d_2, m) = \tau(d_1, \tau(d_2, m))$ and $\tau(1_{D_{\mathbb{C}}}, m) = m$ for all elements d_1, d_2 in the quasitorus $D_{\mathbb{C}}$ and for each point m in the space M.

2 From simple polytopes to complex quasifolds

Let $\mathfrak{d}$ be a real vector space of dimension n, and let Δ be a simple convex polytope of dimension n in the dual space $\mathfrak{d}^*$ (we recall that a polytope of dimension n is simple if there are exactly n edges stemming from each of its vertices). It is our intention to associate to this polytope a family of complex quasifolds, in much the same way that one associates a toric variety to a simple convex polytope that is rational. To do so we follow and extend the procedure for constructing toric varieties as global quotients that is described by Guillemin in [G].

Write the polytope as

$$\Delta = \bigcap_{j=1}^{d} \{\, \mu \in \mathfrak{d}^* \mid \langle \mu, X_j \rangle \geq \lambda_j \,\} \tag{2}$$

for some elements $X_1, \ldots, X_d$ in the vector space $\mathfrak{d}$ and some real numbers $\lambda_1, \ldots, \lambda_d$. Let Q be a quasilattice in the space $\mathfrak{d}$ containing the elements X_j (for example the one that is generated by these elements) and let $\{e_1, \ldots, e_d\}$ denote the standard basis of $\mathbb{R}^d$ and $\mathbb{C}^d$; consider the surjective linear mapping

$$\begin{aligned} \pi\colon \quad \mathbb{R}^d &\longrightarrow \mathfrak{d} \\ e_j &\longmapsto X_j, \end{aligned}$$

and its complexification

$$\begin{aligned} \pi_{\mathbb{C}}\colon \quad \mathbb{C}^d &\longrightarrow \mathfrak{d}_{\mathbb{C}} \\ e_j &\longmapsto X_j. \end{aligned}$$

Consider the quasitorus $\mathfrak{d}/Q$ and its complexification $\mathfrak{d}_{\mathbb{C}}/Q$. The mappings π and $\pi_{\mathbb{C}}$ each induce a group homomorphism,

$$\Pi\colon T^d = \mathbb{R}^d/\mathbb{Z}^d \longrightarrow \mathfrak{d}/Q$$

and

$$\Pi_{\mathbb{C}} : T_{\mathbb{C}}^d = \mathbb{C}^d/\mathbb{Z}^d \longrightarrow \mathfrak{d}_{\mathbb{C}}/Q.$$

We define N to be the kernel of the mapping Π and $N_{\mathbb{C}}$ to be the kernel of the mapping $\Pi_{\mathbb{C}}$. Notice that neither N nor $N_{\mathbb{C}}$ is a torus unless Q is a honest lattice. The mapping $\Pi_{\mathbb{C}}$ defines an isomorphism

$$T_{\mathbb{C}}^d/N_{\mathbb{C}} \longrightarrow \mathfrak{d}_{\mathbb{C}}/Q \tag{3}$$

Remark 2.1 For the complexified group $N_{\mathbb{C}}$ the polar decomposition holds, namely

$$N_{\mathbb{C}} = NA, \tag{4}$$

where $A = \exp(i\mathfrak{n})$. In other words every element $w \in N_{\mathbb{C}}$ can be written *uniquely* as $x \exp(iY)$ where $x \in N$ and $Y \in \mathfrak{n}$. This follows from the definition of N and $N_{\mathbb{C}}$, indeed $N_{\mathbb{C}} = \{\, \exp(Z) \mid Z \in \mathbb{C}^d \quad \text{and} \quad \pi_{\mathbb{C}}(Z) \in Q \,\}$. Write $Z = X + iY$, then $\pi_{\mathbb{C}}(Z) \in Q$ if and only if $\pi(X) \in Q$ and $\pi(Y) = 0$, which implies (4).

Let F denote a codimension-k face of the polytope; this face is defined by a system of k equalities: $\langle \mu, X_j \rangle = \lambda_j$, for $j \in I \subset \{1, \ldots, d\}$. Then we consider the $T_{\mathbb{C}}^d$-orbit $\mathbb{C}_F^d = \{\, (z_1, \cdots, z_d) \in \mathbb{C}^d \mid z_j = 0 \ \text{iff} \ j \in I \,\}$ and we take the union over all the possible faces of the polytope

$$\mathbb{C}_\Delta^d = \bigcup_F \mathbb{C}_F^d.$$

The group $N_{\mathbb{C}}$ acts on the space $\mathbb{C}_\Delta^d$. Consider the space of orbits $\mathbb{C}_\Delta^d/N_{\mathbb{C}}$. We then have:

Theorem 2.2 *Let $\mathfrak{d}$ be a vector space of dimension n, and let $\Delta \subset \mathfrak{d}^*$ be a simple convex polytope. Choose inward-pointing normals to the facets of Δ, $X_1, \ldots, X_d \in \mathfrak{d}$, and let Q be a quasilattice containing these vectors. Then the corresponding quotient space $\mathbb{C}_\Delta^d/N_{\mathbb{C}}$ is a complex quasifold of dimension n. The complex quasitorus $\mathfrak{d}_{\mathbb{C}}/Q$ acts on $\mathbb{C}_\Delta^d/N_{\mathbb{C}}$, this action is holomorphic and has a dense open orbit.*

Before we prove this theorem we need a lemma that will be crucial not only now but also throughout the rest of the article. Consider any face F of the polytope and denote by $S_{\mathbb{C}}^F$ the stabilizer of the $T_{\mathbb{C}}^d$-action on $\mathbb{C}_F^d$ and by $\mathfrak{s}_{\mathbb{C}}^F$ its Lie algebra. Then the fact that the polytope is simple implies that

$$\mathfrak{n}_{\mathbb{C}} \cap \mathfrak{s}_{\mathbb{C}}^F = \{0\}, \tag{5}$$

and this in turn implies that

$$A \cap S_{\mathbb{C}}^F = \{\mathrm{Id}\}, \tag{6}$$

and that the following group is discrete

$$N_{\mathbb{C}} \cap S_{\mathbb{C}}^F = N \cap S_{\mathbb{C}}^F. \tag{7}$$

Equation (5) also implies that π, restricted to $\mathfrak{s}_{\mathbb{C}}^F$, is injective for any face F. In the special case that F is equal to a vertex, μ, we get that $\mathfrak{s}_{\mathbb{C}}^\mu$ is a complement of $\mathfrak{n}_{\mathbb{C}}$ in

$\mathbb{C}^d$, and that the linear mapping $\pi_{\mathbb{C}}$, when restricted to $\mathfrak{s}_{\mathbb{C}}^{\mu}$, defines a (very useful) isomorphism

$$\pi_{\mu} \colon \mathfrak{s}_{\mathbb{C}}^{\mu} \longrightarrow \mathfrak{d}_{\mathbb{C}}. \tag{8}$$

One can then deduce:

Lemma 2.3 *Let μ be a vertex of the polytope Δ, consider the stabilizer $S_{\mathbb{C}}^{\mu}$ of the orbit $\mathbb{C}_{\mu}^d$, its Lie algebra $\mathfrak{s}_{\mathbb{C}}^{\mu}$, and the discrete group $\Gamma_{\mu} = S_{\mathbb{C}}^{\mu} \cap N_{\mathbb{C}}$. Then we have that*
(i) $T_{\mathbb{C}}^d / S_{\mathbb{C}}^{\mu} \simeq N_{\mathbb{C}} / \Gamma_{\mu}$;
(ii) $N_{\mathbb{C}} = \Gamma_{\mu} \exp(\mathfrak{n}_{\mathbb{C}})$;
(iii) *given any complement $\mathfrak{b}_{\mathbb{C}}$ of $\mathfrak{s}_{\mathbb{C}}^{\mu}$ in $\mathbb{C}^d$, we have that*

$$\mathfrak{n}_{\mathbb{C}} = \{\, V - \pi_{\mu}^{-1}(\pi_{\mathbb{C}}(V)) \mid V \in \mathfrak{b}_{\mathbb{C}} \,\}.$$

Proof. (i) Consider the group homomorphism

$$\begin{array}{cccc} \lambda_{\mu} \colon & N_{\mathbb{C}} & \longrightarrow & T_{\mathbb{C}}^d / S_{\mathbb{C}}^{\mu} \\ & n & \longmapsto & [n]. \end{array}$$

Since $\mathfrak{n}_{\mathbb{C}}$ and $\mathfrak{s}_{\mathbb{C}}^{\mu}$ are complementary, we have that $T_{\mathbb{C}}^d = S_{\mathbb{C}}^{\mu} N_{\mathbb{C}}$, therefore λ_{μ} is surjective. The kernel of λ_{μ} is given by Γ_{μ}, so λ_{μ} induces an isomorphism $T_{\mathbb{C}}^d / S_{\mathbb{C}}^{\mu} \simeq N_{\mathbb{C}} / \Gamma_{\mu}$.
(ii) Every element in $N_{\mathbb{C}}$ can be written in the form $\exp(Z)$, where $Z \in \mathbb{C}^d$ is such that $\pi_{\mathbb{C}}(Z) \in Q$. Write now $Z = Z - \pi_{\mu}^{-1}(\pi_{\mathbb{C}}(Z)) + \pi_{\mu}^{-1}(\pi_{\mathbb{C}}(Z))$; it is easy to check that $Z - \pi_{\mu}^{-1}(\pi_{\mathbb{C}}(Z)) \in \mathfrak{n}_{\mathbb{C}}$, and that $\exp(\pi_{\mu}^{-1}(\pi_{\mathbb{C}}(Z))) \in \Gamma_{\mu}$. The group $\Gamma_{\mu} \cap \exp(\mathfrak{n}_{\mathbb{C}})$ is not necessarily trivial, so the decomposition is not necessarily unique.
(iii) Every element of the form $V - \pi_{\mu}^{-1}(\pi_{\mathbb{C}}(V))$, $V \in \mathfrak{b}_{\mathbb{C}}$ clearly belongs to $\mathfrak{n}_{\mathbb{C}}$. Conversely, write every element $Z \in \mathfrak{n}_{\mathbb{C}}$ as $Z = U + V$ according to the decomposition $\mathbb{C}^d = \mathfrak{s}_{\mathbb{C}}^{\mu} \oplus \mathfrak{b}_{\mathbb{C}}$, and notice that $\pi_{\mathbb{C}}(Z) = 0$ implies that $U = -\pi_{\mu}^{-1}(\pi_{\mathbb{C}}(V))$. $\qquad \square$

Proof of Theorem 2.2. We want to define a complex quasifold atlas on the topological space $\mathbb{C}_{\Delta}^d / N_{\mathbb{C}}$. We start by considering a rather special covering of $\mathbb{C}_{\Delta}^d$. To do so we restrict our attention to each vertex μ of the polytope (the corresponding orbit $\mathbb{C}_{\mu}^d$ has the smallest possible dimension $d - n$) and we take the following $T_{\mathbb{C}}^d$-invariant, open and connected neighborhood of the orbit $\mathbb{C}_{\mu}^d$ in $\mathbb{C}^d$:

$$\widehat{V}_{\mu} = \{\, (z_1, \cdots, z_d) \in \mathbb{C}^d \mid z_j \neq 0 \ \text{if} \ j \notin I \,\},$$

where I is the index set corresponding to the vertex μ according to the recipe given below Remark 2.1. Notice that we have $\mathbb{C}_{\Delta}^d = \bigcup_{\mu} \widehat{V}_{\mu}$, where μ ranges over all the vertices of the polytope Δ. Indeed, take $(z_1, \cdots, z_d) \in \mathbb{C}_{\Delta}^d$; then $(z_1, \cdots, z_d) \in \mathbb{C}_G^d$ for some face G, and $(z_1, \cdots, z_d) \in \widehat{V}_{\mu}$ for any vertex μ contained in the face G. The opposite inclusion holds because the polytope is simple. The neighborhoods $\widehat{V}_{\mu}$ are rather special, they are tubular. Let us check this. Fix a point $\underline{z}^{\mu}$ in the orbit $\mathbb{C}_{\mu}^d$, for example $\underline{z}^{\mu} = (z_1^{\mu}, \cdots, z_d^{\mu})$ with

$$\begin{cases} z_j^{\mu} = 0 & \text{if} \quad j \in I \\ z_j^{\mu} = 1 & \text{if} \quad j \notin I. \end{cases} \tag{9}$$

Then there is a $T_{\mathbb{C}}^d$-equivariant biholomorphism given by

$$
\begin{array}{ccc}
T_{\mathbb{C}}^d \times_{S_{\mathbb{C}}^\mu} \tilde{V}_\mu & \longrightarrow & \widehat{V}_\mu \\
[t \quad : \quad \underline{z}] & \longmapsto & t \cdot (\underline{z} + \underline{z}^\mu),
\end{array}
$$

where

$$
\tilde{V}_\mu = \prod_{i \in I} \mathbb{C}e_i \simeq \mathbb{C}^n
$$

is the holomorphic slice at $\underline{z}^\mu$ for the $T_{\mathbb{C}}^d$-action on $\mathbb{C}_\Delta^d$ and

$$
S_{\mathbb{C}}^\mu = \{\, (z_1, \cdots, z_d) \in T_{\mathbb{C}}^d \mid z_j = 1 \ \text{if} \ j \notin I \,\}
$$

is the stabilizer of the action at $\underline{z}^\mu$. Notice that $\tilde{V}_\mu = \mathfrak{s}_{\mathbb{C}}^\mu$. We now prove that the subsets $V_\mu = \widehat{V}_\mu / N_{\mathbb{C}}$ are complex charts for the quotient space $\mathbb{C}_\Delta^d / N_{\mathbb{C}}$. Consider the continuous surjective mapping

$$
\begin{array}{cccc}
p_\mu: & \tilde{V}_\mu & \longrightarrow & V_\mu \\
& \underline{z} & \longmapsto & [\underline{z}^\mu + \underline{z}].
\end{array}
$$

The discrete group $\Gamma_\mu = N_{\mathbb{C}} \cap S_{\mathbb{C}}^\mu$ acts on the set $\tilde{V}_\mu$ as follows:

$$
\begin{array}{ccc}
\Gamma_\mu \times \tilde{V}_\mu & \longrightarrow & \tilde{V}_\mu \\
(t \quad, \quad \underline{z}) & \longmapsto & t \cdot \underline{z}.
\end{array}
$$

This action is holomorphic, free on a connected dense open set and the mapping p_μ induces a continuous bijection

$$
\phi_\mu: \tilde{V}_\mu / \Gamma_\mu \longrightarrow V_\mu.
$$

To show that ϕ_μ is actually a homeomorphism, we only need to show that it is an open mapping. This amounts to showing that, given a Γ_μ-invariant open subset W of $\tilde{V}_\mu$, then $N_{\mathbb{C}} \cdot (\underline{z}^\mu + W)$ is an open subset of $\mathbb{C}_\Delta^d$. Since W is Γ_μ-invariant, by Lemma 2.3, (ii), we have that $N_{\mathbb{C}} \cdot (\underline{z}^\mu + W) = \exp(\mathfrak{n}_{\mathbb{C}}) \cdot (\underline{z}^\mu + W)$. Applying Lemma 2.3, (iii), we construct the surjective mapping

$$
\begin{array}{ccc}
W \times \Pi_{j \notin I} \mathbb{C}e_j & \longrightarrow & \exp(\mathfrak{n}_{\mathbb{C}})(\underline{z}^\mu + W) \\
(\underline{w} \quad, \quad V) & \longmapsto & \exp(V) \cdot \underline{z}^\mu + \exp(-\pi_\mu^{-1}(\pi_{\mathbb{C}}(V))) \cdot \underline{w}.
\end{array}
$$

The determinant of its Jacobian matrix is given by $\sum_{j=1}^n e^{K_j(Y)} + 4\pi^2 \sum_{h \notin I} e^{-4\pi Y_h}$, where $Y = \frac{i}{2}(\overline{V} - V)$ and the K_j's are real linear functions. This implies, by the inverse function theorem, that $\exp(\mathfrak{n}_{\mathbb{C}})(\underline{z}^\mu + W)$ is an open subset of $\mathbb{C}_\Delta^d$.

Let us now show that these charts are compatible. Consider two vertices of Δ, μ and ν, and let I and J be the corresponding subsets of $\{1, \ldots, d\}$. Assume that the corresponding charts, V_μ and V_ν, have non-empty intersection, and consider the two sets $W_\mu = \phi_\mu^{-1}(V_\mu \cap V_\nu)$ and $W_\nu = \phi_\nu^{-1}(V_\mu \cap V_\nu)$. We want to describe these two sets as submodels of $\tilde{V}_\mu / \Gamma_\mu$ and $\tilde{V}_\nu / \Gamma_\nu$ respectively, and show that $g_{\mu\nu} = \phi_\nu^{-1} \circ \phi_\mu$

is a biholomorphism of models. Consider $\tilde{W}_\mu = p_\mu^{-1}(\phi_\mu^{-1}(V_\mu \cap V_\nu)) \subset \tilde{V}_\mu$ and $\tilde{W}_\nu = p_\nu^{-1}(\phi_\nu^{-1}(V_\mu \cap V_\nu)) \subset \tilde{V}_\nu$. Now,

$$\tilde{W}_\mu = \left(\prod_{j \in I \cap J} \mathbb{C} e_j \right) \times \left(\prod_{j \in I \setminus I \cap J} \mathbb{C}^* e_j \right)$$

and

$$\tilde{W}_\nu = \left(\prod_{j \in I \cap J} \mathbb{C} e_j \right) \times \left(\prod_{j \in J \setminus I \cap J} \mathbb{C}^* e_j \right).$$

Consider the universal covers $W_\mu^\#$, $W_\nu^\#$ of $\tilde{W}_\mu$, $\tilde{W}_\nu$ respectively. Notice that we have that

$$W_\mu^\# = \left(\prod_{j \in I \cap J} \mathbb{C} e_j \right) \times \left(\prod_{j \in I \setminus I \cap J} \mathbb{C} e_j \right) \simeq \prod_{j \in I} \mathbb{C} e_j = \tilde{V}_\mu$$

and

$$W_\nu^\# = \left(\prod_{j \in I \cap J} \mathbb{C} e_j = \right) \times \left(\prod_{j \in J \setminus I \cap J} \mathbb{C} e_j \right) \simeq \prod_{j \in J} \mathbb{C} e_j = \tilde{V}_\nu,$$

with projection maps

$$\begin{array}{ccc} W_\mu^\# & \longrightarrow & \tilde{W}_\mu \\ (\underline{z}, \underline{\zeta}) & \longmapsto & (\underline{z}, \exp \underline{\zeta}) \end{array}$$

and

$$\begin{array}{ccc} W_\nu^\# & \longrightarrow & \tilde{W}_\nu \\ (\underline{u}, \underline{\eta}) & \longmapsto & (\underline{u}, \exp \underline{\eta}). \end{array}$$

Consider the discrete groups

$$\Lambda_\mu = \left\{ (\exp Z, W) \mid Z \in \bigoplus_{j \in I \cap J} \mathbb{C} e_j, \ W \in \bigoplus_{j \in I \setminus I \cap J} \mathbb{C} e_j, \ \pi_\mathbb{C}(Z + W) \in Q \right\}$$

and

$$\Lambda_\nu = \left\{ (\exp U, V) \mid U \in \bigoplus_{j \in I \cap J} \mathbb{C} e_j, \ V \in \bigoplus_{j \in J \setminus I \cap J} \mathbb{C} e_j, \ \pi_\mathbb{C}(U + V) \in Q \right\}$$

acting respectively on $W_\mu^\#$, $W_\nu^\#$ as follows

$$\begin{array}{ccc} (\Lambda_\mu & , & W_\mu^\#) & \longrightarrow & W_\mu^\# \\ ((\exp Z, W) & , & (\underline{z}, \underline{\zeta})) & \longmapsto & (\exp Z \cdot \underline{z}, W + \underline{\zeta}) \end{array}$$

$$\begin{array}{ccc} (\Lambda_\nu & , & W_\nu^\#) & \longrightarrow & W_\nu^\# \\ ((\exp U, V) & , & (\underline{u}, \underline{\eta})) & \longmapsto & (\exp U \cdot \underline{u}, V + \underline{\eta}). \end{array}$$

Notice that the projections induce homeomorphisms $W_\mu^\# / \Lambda_\mu \simeq W_\mu$ and $W_\nu^\# / \Lambda_\nu \simeq W_\nu$. Now we want to show that there is an equivariant biholomorphism $g_{\mu\nu}^\# : W_\mu^\# \longrightarrow W_\nu^\#$

that projects down to the mapping $g_{\mu\nu}$. Consider the isomorphisms π_μ and π_ν defined by (8). Notice that $\pi_\nu^{-1} \cdot \pi_\mu$ defines a biholomorphism from $W_\mu^\#$ to $W_\nu^\#$ that is equal to the identity on $\prod_{j \in I \cap J} \mathbb{C}e_j$. Let moreover ρ, respectively σ, denote the projection of $W_\nu^\#$ onto the factor $\prod_{j \in I \cap J} \mathbb{C}e_j$, respectively $\prod_{j \in J \setminus I \cap J} \mathbb{C}e_j$. Define

$$
g_{\mu\nu}^\#\colon \quad W_\mu^\# \quad \longrightarrow \quad W_\nu^\#
$$
$$
(\underline{z}, \underline{\zeta}) \quad \longmapsto \quad \left(\exp\left(\rho\left(\pi_\nu^{-1} \circ \pi_\mu(\underline{\zeta}) \right) \right) \cdot \underline{z}, \sigma\left(\pi_\nu^{-1} \circ \pi_\mu(\underline{\zeta}) \right) \right).
$$

It is straightforward to check that $g_{\mu\nu}^\#$ is an equivariant biholomorphism that projects down to $g_{\mu\nu}$. Now complete this collection of charts with all the other compatible charts. This concludes the proof that $\mathbb{C}_\Delta^d/N_\mathbb{C}$ is a complex quasifold of dimension n. The standard holomorphic action of $T_\mathbb{C}^d$ on $\mathbb{C}_\Delta^d$ induces a holomorphic action of $T_\mathbb{C}^d/N_\mathbb{C}$ on $\mathbb{C}_\Delta^d/N_\mathbb{C}$. The isomorphism (3) allows us to define a holomorphic action $\tau\colon (\mathfrak{d}_\mathbb{C}/Q) \times (\mathbb{C}_\Delta^d/N_\mathbb{C}) \to \mathbb{C}_\Delta^d/N_\mathbb{C}$. To check that τ is holomorphic, consider, for each vertex μ of the polytope, the chart V_μ, and the bijection π_μ given in (8). We then have that the following diagram commutes

$$
\begin{array}{ccc}
\mathfrak{d}_\mathbb{C} \times \tilde{V}_\mu & \xrightarrow{\ \tilde{\tau}_\mu\ } & \tilde{V}_\mu \\
(W, \underline{z}) & \longmapsto & \exp(\pi_\mu^{-1}(W)) \cdot \underline{z} \\
\downarrow & & \downarrow \\
(\mathfrak{d}_\mathbb{C}/Q) \times (\tilde{V}_\mu/\Gamma_\mu) & \xrightarrow{\ \tau_\mu\ } & \tilde{V}_\mu/\Gamma_\mu \\
([W], [\underline{z}]) & \longmapsto & [\exp(\pi_\mu^{-1}(W)) \cdot \underline{z}] \\
\downarrow & & \downarrow \\
(\mathfrak{d}_\mathbb{C}/Q) \times V_\mu & \xrightarrow{\ \tau\ } & V_\mu \\
([W], [\underline{z} + \underline{z}_\mu]) & \longmapsto & [\exp(\pi_\mu^{-1}(W)) \cdot (\underline{z} + \underline{z}_\mu)] = [\exp(\pi_\mu^{-1}(W)) \cdot \underline{z} + \underline{z}_\mu]
\end{array}
$$

and that $\tilde{\tau}_\mu$ is a holomorphic mapping. Notice finally that the dense open orbit for this action is given by $\mathbb{C}_F^d/N_\mathbb{C}$, where $F = \mathrm{Int}(\Delta)$. $\qquad\square$

Remark 2.4 Suppose that the polytope Δ is rational with respect to a lattice $L \subset \mathfrak{d}$. Choose inward-pointing normal vectors to the facets of Δ that are primitive in L, and take $Q = L$. Then the corresponding quotient $\mathbb{C}_\Delta^d/N_\mathbb{C}$ is a complex orbifold and contains a dense open orbit for the action of the honest torus $T_\mathbb{C} = \mathfrak{d}_\mathbb{C}/L$; it is the usual toric variety associated to the polytope Δ.

Remark 2.5 Notice that, just like for honest toric varieties, the quasitorus $\mathfrak{d}_\mathbb{C}/Q$ is contained in $\mathbb{C}_\Delta^d/N_\mathbb{C}$ as a dense open subset, and the action of $\mathfrak{d}_\mathbb{C}/Q$ on $\mathbb{C}_\Delta^d/N_\mathbb{C}$ extends the standard action of $\mathfrak{d}_\mathbb{C}/Q$ on itself.

Let us test this construction on three examples. Only the third is an example of a truly non-rational polytope.

Example 2.6 (The unit interval) Let us consider the polytope $[0, 1] \subset \mathbb{R}^*$ and let us apply the above construction to the choice of vectors $X_1 = s$, $X_2 = -t$, with $s, t > 0$, $s/t \notin \mathbb{Q}$, and to the choice of quasilattice $Q = s\mathbb{Z} + t\mathbb{Z}$. It is easy to check that $\mathbb{C}^2_\Delta = \mathbb{C}^2 \setminus \{0\}$ and that $N_\mathbb{C} = \{ (e^{2\pi i Z}, e^{2\pi i \frac{s}{t} Z}) \mid Z \in \mathbb{C} \}$. We cover $\mathbb{C}^2_\Delta / N_\mathbb{C}$ with two charts, one for each of its vertices, $\mathrm{S} = \{0\}$ and $\mathrm{N} = \{1\}$: $V_\mathrm{S} = \left\{ [z_1 : z_2] \in \mathbb{C}^2_\Delta / N_\mathbb{C} \mid z_2 \neq 0 \right\}$ and $V_\mathrm{N} = \left\{ [z_1 : z_2] \in \mathbb{C}^2_\Delta / N_\mathbb{C} \mid z_1 \neq 0 \right\}$. Consider now the the discrete group $\Gamma_\mathrm{S} = \mathbb{Z}$ acting on the set $\tilde{V}_\mathrm{S} = \mathbb{C}$ according to the rule $(k, z) \mapsto e^{2\pi i \frac{t}{s} k} \cdot z$; this action is holomorphic and free on the connected, dense subset $\tilde{V}_\mathrm{S} \setminus \{0\}$ and the mapping

$$\begin{aligned} \phi_\mathrm{S}: \quad \tilde{V}_\mathrm{S}/\Gamma_\mathrm{S} &\longrightarrow \quad V_\mathrm{S} \\ [z] &\longmapsto \quad [z : 1] \end{aligned}$$

is a homeomorphism. Similarly the group $\Gamma_\mathrm{N} = \mathbb{Z}$ acts on $\tilde{V}_\mathrm{N} = \mathbb{C}$ by the rule $(m, z) \mapsto e^{2\pi i \frac{s}{t} m} \cdot w$; this action is holomorphic and free on the connected, dense subset $\tilde{V}_\mathrm{N} \setminus \{0\}$ and the mapping

$$\begin{aligned} \phi_\mathrm{N}: \quad \tilde{V}_\mathrm{N}/\Gamma_\mathrm{N} &\longrightarrow \quad V_\mathrm{N} \\ [w] &\longmapsto \quad [1 : w] \end{aligned}$$

is a homeomorphism. Let us show that these two charts are compatible. The set $\phi_\mathrm{S}^{-1}(V_\mathrm{S} \cap V_\mathrm{N})$ is a submodel of $\tilde{V}_\mathrm{S}/\Gamma_\mathrm{S}$: it is the quotient of the space $W_\mathrm{S}^\# = \mathbb{C}$ by the action of $\Lambda_\mathrm{S} = \mathbb{Z}^2$ given by $(h, k) \cdot \zeta = \zeta + h + \frac{t}{s} k$. Similarly the set $\phi_\mathrm{N}^{-1}(V_\mathrm{S} \cap V_\mathrm{N})$ is the quotient of the space $W_\mathrm{N}^\# = \mathbb{C}$ by the action of $\Lambda_\mathrm{N} = \mathbb{Z}^2$ given by $(l, m) \cdot \eta = \eta + l + \frac{s}{t} m$. The bijective mapping

$$\begin{aligned} g_\mathrm{SN} = \phi_\mathrm{N}^{-1} \circ \phi_\mathrm{S} : \phi_\mathrm{S}^{-1}(V_\mathrm{S} \cap V_\mathrm{N}) &\longrightarrow \quad \phi_\mathrm{N}^{-1}(V_\mathrm{S} \cap V_\mathrm{N}) \\ [z] &\longmapsto \quad \left[w = z^{-\frac{s}{t}} \right] \end{aligned}$$

is a biholomorphism of models: its lift is given by $\zeta \longmapsto \eta = -\frac{s}{t} \zeta$. Now complete this collection with all the other compatible charts.

Remark 2.7 We can see from the previous example that complex quasifolds corresponding to the same polytope Δ are not in general biholomorphic, in fact they are not even diffeomorphic. This was also visible in the rational case: the same construction applied to s, t relatively prime integers yields in fact either ordinary or weighted projective space, which have different complex orbifold structures. This may appear to be in contradiction with Theorem 9.4 of Lerman-Tolman [LT], which implies that these spaces are biholomorphic - in reality their notion of biholomorphism is algebraic and, unlike ours, does not keep track of the orbifold structure.

Example 2.8 (The right triangle) Consider the right triangle in $(\mathbb{R}^2)^*$ of vertices $(0, 0)$, $(s, 0)$ and $(0, t)$, where s, t are two positive real numbers such that $s/t \notin \mathbb{Q}$. We apply the construction to the choice of inward-pointing normals $X_1 = (1, 0)$, $X_2 = (0, 1)$, $X_3 = (-t, -s)$ and to the quasilattice $Q = X_1 \mathbb{Z} + X_2 \mathbb{Z} + X_3 \mathbb{Z}$. Then we have that $\mathbb{C}^3_\Delta = \mathbb{C}^3 \setminus \{0\}$ and $N_\mathbb{C} = \{ (e^{2\pi i t Z}, e^{2\pi i s Z}, e^{2\pi i Z}) \mid Z \in \mathbb{C} \}$. We cover $\mathbb{C}^3_\Delta / N_\mathbb{C}$

with three charts, one for each of its vertices: $V_1 = \left\{ [z_1 : z_2 : z_3] \in \mathbb{C}^3_\Delta/N_\mathbb{C} \mid z_1 \neq 0 \right\}$, $V_2 = \left\{ [z_1 : z_2 : z_3] \in \mathbb{C}^3_\Delta/N_\mathbb{C} \mid z_2 \neq 0 \right\}$ and $V_3 = \left\{ [z_1 : z_2 : z_3] \in \mathbb{C}^3_\Delta/N_\mathbb{C} \mid z_3 \neq 0 \right\}$. The discrete group $\Gamma_1 = \mathbb{Z}$ acts on $\tilde{V}_1 = \mathbb{C}^2$ according to the rule $h \cdot (z_2, z_3) = (e^{2\pi i \frac{s}{t}h} \cdot z_2, e^{2\pi i \frac{1}{t}h} \cdot z_3)$ and the mapping

$$\begin{array}{cccc} \phi_1 : & \tilde{V}_1/\Gamma_1 & \longrightarrow & V_1 \\ & [z_2, z_3] & \longmapsto & [1 : z_2 : z_3] \end{array}$$

is a homeomorphism. Similarly the group $\Gamma_2 = \mathbb{Z}$ acts on $\tilde{V}_2 = \mathbb{C}^2$ by the rule $k \cdot (z_1, z_3) = (e^{2\pi i \frac{t}{s}k} \cdot z_1, e^{2\pi i \frac{1}{s}k} \cdot z_3)$ and the mapping

$$\begin{array}{cccc} \phi_2 : & \tilde{V}_2/\Gamma_2 & \longrightarrow & V_2 \\ & [z_1, z_3] & \longmapsto & [z_1 : 1 : z_3] \end{array}$$

is a homeomorphism. Finally the group $\Gamma_3 = \mathbb{Z}$ acts on $\tilde{V}_3 = \mathbb{C}^2$ according to the rule $l \cdot (z_1, z_2) = (e^{2\pi i t l} \cdot z_1, e^{2\pi i s l} \cdot z_2)$ and the mapping

$$\begin{array}{cccc} \phi_3 : & \tilde{V}_3/\Gamma_3 & \longrightarrow & V_3 \\ & [z_1, z_2] & \longmapsto & [z_1 : z_2 : 1] \end{array}$$

is a homeomorphism. The changes of charts work as in the previous example.

Example 2.9 (The regular pentagon) Let us take the regular pentagon in $(\mathbb{R}^2)^*$ of vertices $(1, 0,)$, (a, b), (c, d), $(c, -d)$ and $(a, -b)$, where $a = \cos \frac{2\pi}{5}$, $b = \sin \frac{2\pi}{5}$, $c = \cos \frac{4\pi}{5}$, $d = \sin \frac{4\pi}{5}$. There exists no lattice L with respect to which this simple convex polytope is rational. Let us apply the above construction to the choice of inward-pointing normal vectors $X_1 = (1, 0)$, $X_2 = (a, b)$, $X_3 = (c, d)$, $X_4 = (c, -d)$, $X_5 = (a, -b)$ and to the choice of quasilattice $Q = \sum_{j=1}^5 X_j \mathbb{Z}$. Then we have that

$$N_\mathbb{C} = \left\{ \left(e^{2\pi i Z_1}, e^{2\pi i Z_2}, e^{2\pi i Z_3}, e^{2\pi i [2a(Z_2 - Z_3) + Z_1]}, e^{2\pi i [2a(Z_2 - Z_1) + Z_3)]} \right) \mid (Z_1, Z_2, Z_3) \in \mathbb{C}^3 \right\}.$$

We cover the quasifold $\mathbb{C}^5_\Delta/N_\mathbb{C}$ with five charts:

$$V_1 = \left\{ [z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{C}^5_\Delta/N_\mathbb{C} \mid z_1, z_2, z_3 \neq 0 \right\},$$

$$V_2 = \left\{ [z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{C}^5_\Delta/N_\mathbb{C} \mid z_2, z_3, z_4 \neq 0 \right\},$$

$$V_3 = \left\{ [z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{C}^5_\Delta/N_\mathbb{C} \mid z_3, z_4, z_5 \neq 0 \right\},$$

$$V_4 = \left\{ [z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{C}^5_\Delta/N_\mathbb{C} \mid z_1, z_4, z_5 \neq 0 \right\}$$

and

$$V_5 = \left\{ [z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{C}^5_\Delta/N_\mathbb{C} \mid z_1, z_2, z_5 \neq 0 \right\}.$$

The discrete group $\Gamma_1 = \mathbb{Z}^3$ acts on the set $\tilde{V}_1 = \mathbb{C}^2$ according to the rule $(h, k, l) \cdot (z_4, z_5) = (e^{4\pi i a(k-l)} \cdot z_4, e^{4\pi i a(k-h)} \cdot z_5)$ and the mapping

$$\begin{array}{cccc} \phi_1 : & \tilde{V}_1/\Gamma_1 & \longrightarrow & V_1 \\ & [z_4 : z_5] & \longmapsto & [1 : 1 : 1 : z_4 : z_5] \end{array}$$

is a homeomorphism. We leave the discussion of the other four charts, and their mutual compatibility, to the reader.

3   Kähler structures

In the previous section we associated to any simple convex polytope $\Delta \subset \mathfrak{d}^*$, together with a choice of normal vectors X_j, $j = 1, \ldots, d$, and of a quasilattice $Q \subset \mathfrak{d}$ containing the X_j's, a complex quasifold of dimension n. Starting with the same data it is possible to construct a symplectic quasifold of dimension $2n$, endowed with an effective Hamiltonian action of the quasitorus $\mathfrak{d}/Q$, such that the image of the corresponding moment mapping is exactly Δ (see [P, Theorem 3.3]). The complex and symplectic quasifolds are both described as orbit spaces, the first is the quotient of $\mathbb{C}^d_\Delta$ by $N_{\mathbb{C}}$, the second is a symplectic quotient with respect to the action of N on $\mathbb{C}^d$. We want to prove that the complex and symplectic quotient can be identified, according to a general principle initiated by Kempf-Ness [KN] and later developed by Kirwan [K] and Ness [N]. More precisely, we shall prove that the two quotients are diffeomorphic and that the complex and symplectic structure are compatible, and thus define the structure of a Kähler quasifold (see Definition 1.11).

To begin with, let us briefly recall from [P] the construction of the symplectic quotient that we are interested in. Consider the mapping $J(\underline{z}) = \sum_{j=1}^{d}(|z_j|^2 + \lambda_j)e_j^*$, where the λ_j's are given in (2) and are uniquely determined by our choice of normal vectors. The mapping J is a moment mapping for the standard action of T^d on $\mathbb{C}^d$. Consider now the subgroup $N \subset T^d$ and the corresponding inclusion of Lie algebras $\iota \colon \mathfrak{n} \to \mathbb{R}^d$. The mapping $\Psi \colon \mathbb{C}^d \to \mathfrak{n}^*$ given by $\Psi = \iota^* \circ J$ is a moment mapping for the induced action of N on $\mathbb{C}^d$. Then the quotient space $\Psi^{-1}(0)/N$ is a compact symplectic quasifold of dimension $2n$; the quasitorus $\mathfrak{d}/Q$ acts on $\Psi^{-1}(0)/N$ in an effective, Hamiltonian fashion, and the image of the corresponding moment map is the polytope Δ. Let us now define a mapping between the symplectic and complex quotient. To define this mapping and to show that it is bijective we adapt to our setting the method described by Guillemin in [G, Appendix 1] for the smooth case. Before we can define this mapping, we need a preliminary lemma.

Lemma 3.1 *The zero set $\Psi^{-1}(0)$ is contained in $\mathbb{C}^d_\Delta$. Moreover, for any face F of the polytope Δ, the orbit $\mathbb{C}^d_F$ intersects $\Psi^{-1}(0)$ in at least one point.*

Proof. We summarize the proof given in [G, page 115], which goes through without modification. Consider the exact sequence

$$0 \longrightarrow \mathfrak{d}^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} (\mathfrak{n})^* \longrightarrow 0, \tag{10}$$

where π is the projection defined in (2). Notice that $\underline{z} \in \Psi^{-1}(0)$ if, and only if $\iota^*(J(\underline{z})) = 0$. By (10) we have that, for a given $\underline{z} \in \Psi^{-1}(0)$, there exists a unique $\zeta \in \mathfrak{d}^*$ such that $J(\underline{z}) = \pi^*(\zeta)$. By making use of the explicit expression of J we find

$$|z_j|^2 = \langle \zeta, X_j \rangle - \lambda_j, \qquad j = 1, \ldots, d. \tag{11}$$

This implies that, given $\underline{z} \in \Psi^{-1}(0)$, the corresponding ζ lies in Δ and that, again by (11), $\underline{z}$ is in $\mathbb{C}^d_F$, where F is the face of Δ containing ζ in its interior. Similarly, given any face F of the polytope, we can always find a point $\underline{z} \in \Psi^{-1}(0) \cap \mathbb{C}^d_F$.   □

By Lemma 3.1, there is an injection

$$\mathcal{I} \colon \Psi^{-1}(0) \longrightarrow \mathbb{C}^d_\Delta, \tag{12}$$

which induces the mapping

$$\chi \colon \Psi^{-1}(0)/N \longrightarrow \mathbb{C}^d_\Delta/N_\mathbb{C} \tag{13}$$

that sends an N-orbit to the corresponding $N_\mathbb{C}$-orbit. This mapping is equivariant with respect to the actions of the quasitori $\mathfrak{d}/Q$ and $\mathfrak{d}_\mathbb{C}/Q$. We are now ready to state the main result of this section.

Theorem 3.2 *Let $\mathfrak{d}$ be a vector space of dimension n, and let $\Delta \subset \mathfrak{d}^*$ be a simple convex polytope. Choose inward-pointing normals to the facets of Δ, $X_1, \ldots, X_d \in \mathfrak{d}$, and let Q be a quasilattice containing these vectors. Then the mapping*

$$\chi \colon \Psi^{-1}(0)/N \longrightarrow \mathbb{C}^d_\Delta/N_\mathbb{C}$$

is an equivariant diffeomorphism of quasifolds. Moreover the induced symplectic form on the complex quasifold $\mathbb{C}^d_\Delta/N_\mathbb{C}$ is Kähler.

For the definitions of symplectic quasifold, diffeomorphism of quasifolds, and pullback of a differential form, we refer the reader to [P]. Before we can proceed with the proof of this theorem we need a number of remarks and lemmas.

Remark 3.3 The group $A = \exp(i\mathfrak{n})$ moves any point $\underline{z} \in \Psi^{-1}(0)$ out of $\Psi^{-1}(0)$. To see this, remark that, for any vector $Y \in \mathfrak{n}$, the corresponding function Ψ_Y satisfies $d\Psi_Y = \imath_{\tilde{Y}}\omega_0$, where $\omega_0 = \frac{1}{2\pi i} \sum_{j=1}^d dz_j \wedge d\bar{z}_j$ is the standard symplectic form of $\mathbb{C}^d$ and $\tilde{Y}$ is the vector field generated by Y. This implies that the gradient of the function Ψ_Y is the vector field generated by iY, which by (6) does not vanish in $\mathbb{C}^d_\Delta$. The function Ψ_Y is therefore strictly increasing along its gradient flow, hence $\exp(tiY) \cdot \underline{z} \notin \Psi^{-1}(0)$, for all $Y \in \mathfrak{n}$ and all $t \neq 0$.

Lemma 3.4 *The following facts are equivalent:*
(i) the mapping χ is bijective;
(ii) every $N_\mathbb{C}$-orbit in $\mathbb{C}^d_\Delta$ intersects $\Psi^{-1}(0)$ in an N-orbit;
(iii) every A-orbit in $\mathbb{C}^d_\Delta$ intersects $\Psi^{-1}(0)$ in at least one point.

Proof. It is obvious that (i) is equivalent to (ii). Before going on with the proof let us re-state the second and third point:
(ii) for each $N_\mathbb{C}$-orbit through a point $\underline{z} \in \mathbb{C}^d_\Delta$ there exists a point $\underline{u} \in \Psi^{-1}(0)$ such that $(N_\mathbb{C} \cdot \underline{z}) \cap \Psi^{-1}(0) = N \cdot \underline{u}$;
(iii) for each A-orbit through a point $\underline{z} \in \mathbb{C}^d_\Delta$ there exists at least one point $\underline{w} \in (A \cdot \underline{z}) \cap \Psi^{-1}(0)$.
    The proof is based on three key facts: the polar decomposition for the group $N_\mathbb{C}$ (see Remark 2.1), the invariance of $\Psi^{-1}(0)$ under the action of N, and Remark 3.3. To see that (ii) implies (iii), consider an A-orbit through $\underline{z} \in \mathbb{C}^d_\Delta$. Then, by (ii), there

exists $\underline{u} \in \Psi^{-1}(0)$ such that $(N_{\mathbb{C}} \cdot \underline{z}) \cap \Psi^{-1}(0) = N \cdot \underline{u}$. Since $N_{\mathbb{C}} = NA$, there exists $\underline{w} \in N \cdot \underline{u} \subset \Psi^{-1}(0)$ such that $\underline{w} \in A \cdot \underline{z}$. To see that (iii) implies (ii), consider any $N_{\mathbb{C}}$-orbit through $\underline{z}$. By assumption, the orbit $A \cdot \underline{z}$ intersects $\Psi^{-1}(0)$ in at least one point $\underline{w}$. Since $N_{\mathbb{C}} = NA$ we have that $N_{\mathbb{C}} \cdot \underline{z} = N_{\mathbb{C}} \cdot \underline{w}$, that $N \cdot \underline{w} \subset (N_{\mathbb{C}} \cdot \underline{z}) \cap \Psi^{-1}(0)$, and that this inclusion is in fact an equality by Remark 3.3. $\qquad\square$

Remark 3.5 If there exists a point $\underline{w} \in (A \cdot \underline{z}) \cap \Psi^{-1}(0)$, then $\underline{w}$ is uniquely determined by this property. If $\underline{w}$ and $\underline{w}'$ are in $(A \cdot \underline{z}) \cap \Psi^{-1}(0)$, then there exists an $a \in A$ such that $a \cdot \underline{w} = \underline{w}'$. By Remark 3.3 we have that a is the identity and that $\underline{w} = \underline{w}'$.

The following result will be essential to proving that the mapping χ is a bijection.

Lemma 3.6 *The moment mapping Ψ maps any A-orbit diffeomorphically onto an open convex cone in $\mathfrak{n}^*$; moreover, if two A-orbits lie in the same $T_{\mathbb{C}}^d$-orbit $\mathbb{C}_F^d \subset \mathbb{C}_\Delta^d$, then their images with respect to Ψ are identical.*

Proof. The proof given in [G, Appendix 1: Theorem 2.1 and Theorem 2.2] applies, the only delicate point here is to notice that, by (6), A still acts freely on $\mathbb{C}_\Delta^d$. Notice also that, by (7), this is certainly not true for the action of N. We now want to study the image by Ψ of an A-orbit through a point $\underline{z} \in \mathbb{C}_\Delta^d$, namely

$$\Psi(A \cdot \underline{z}) = \left\{ \Psi\left(e^{-2\pi\alpha_1(X)} \cdot z_1, \ldots, e^{-2\pi\alpha_d(X)} \cdot z_d \right) \mid X \in \mathfrak{n} \right\},$$

where $\alpha_j = \iota^*(e_j^*)$, $j = 1, \ldots, d$. Since A acts freely on $\mathbb{C}_\Delta^d$, the exponential mapping defines a diffeomorphism between $\mathfrak{n}$ and $A \cdot z$, and we can identify the A-orbit with $\mathfrak{n}$. Therefore the set we are interested in is the image of the mapping

$$
\begin{array}{rccc}
f: & \mathfrak{n} & \longrightarrow & \mathfrak{n}^* \\
& X & \longmapsto & \Psi\left(e^{-2\pi\alpha_1(X)} \cdot z_1, \ldots, e^{-2\pi\alpha_d(X)} \cdot z_d \right).
\end{array}
$$

The point $\underline{z}$ lies in $\mathbb{C}_F^d$ for a face F of the polytope Δ. Let I be the corresponding set of indices, as defined in Section 2, then

$$f(X) = \sum_{j \notin I} e^{-4\pi\alpha_j(X)} |z_j|^2 \alpha_j + \lambda_j \alpha_j.$$

Now we only have to check that

$$\operatorname{span}\{\alpha_j \in \mathfrak{n}^* \mid j \notin I\} = \mathfrak{n}^*; \tag{14}$$

this is equivalent to saying that there does not exist a non zero $X \in \mathfrak{n}$ such that $\alpha_j(X) = 0$ for all $j \notin I$. Indeed, if there was such an X, it would lie in $\mathfrak{n} \cap \mathfrak{s}_{\mathbb{C}}^F$ which is $\{0\}$. Using (14) one can prove, following [G], that f maps $\mathfrak{n}$ diffeomorphically into an open convex cone of $\mathfrak{n}^*$, and also that the image of f depends only on I. $\qquad\square$

We are now ready to proceed with the proof of the main result of this section.

Proof of Theorem 3.2. The proof is divided into several steps. We first prove that the mapping χ is bijective and continuous. Then we lift χ locally to prove that it is a

diffeomorphism, and finally we show that the pull-back via χ^{-1} of the symplectic form of $\Psi^{-1}(0)/N$, is Kähler on the complex quasifold $\mathbb{C}_\Delta^d/N_{\mathbb{C}}$.

In order to prove that χ is bijective we will show that Lemma 3.6 implies that, for a given A-orbit through a point $\underline{z} \in \mathbb{C}_\Delta^d$, there exists a point $\underline{w} \in \Psi^{-1}(0)$ such that $(A \cdot \underline{z}) \cap \Psi^{-1}(0) = \{\underline{w}\}$. Then we will apply Lemma 3.4 to conclude. To prove the existence of $\underline{w}$, consider the $T_{\mathbb{C}}^d$-orbit $\mathbb{C}_F^d$ containing $A \cdot \underline{z}$. By Lemma 3.1 this $\mathbb{C}_F^d$ intersects $\Psi^{-1}(0)$ in at least one point, $\underline{u}$. On the other hand $\Psi(A \cdot \underline{z}) = \Psi(A \cdot \underline{u}) \subset \mathfrak{n}^*$ by Lemma 3.6, and the common image contains $0 = \Psi(\underline{u})$. So there exists $a \in A$ such that $\Psi(a \cdot \underline{z}) = 0$; then $\underline{w} = a \cdot \underline{z} \in (A \cdot \underline{z}) \cap \Psi^{-1}(0)$. Notice that, by Remark 3.5, the point $\underline{w}$ is uniquely determined. Continuity of the mapping χ is implied by that of the immersion $\mathcal{I}$.

For the next step we need a covering of the symplectic quotient by a collection of its charts. We refer the reader to [P, Theorem 3.1] for the construction of the full atlas. Let μ be a vertex of the polytope Δ, and let $I = \{r_1, \dots, r_n\} \subset \{1, \dots, d\}$ denote the corresponding subset of indices. For each vertex of Δ we want to construct a corresponding chart. Let $(a_{jh}) \in M_{(n,d)}(\mathbb{R})$ denote the matrix of the mapping $\pi \colon \mathbb{R}^d \to \mathfrak{d}$ with respect to the basis $\{X_j \,|\, j \in I\}$ of $\mathfrak{d}$ and the standard basis of $\mathbb{R}^d$. Consider the open subset of $\tilde{V}_\mu$ defined as follows

$$\tilde{U}_\mu = \left\{ \underline{w} \in \tilde{V}_\mu \;|\; \sum_{j=1}^n a_{jh}\left(|w_{r_j}|^2 + \lambda_{r_j}\right) - \lambda_h > 0, \; h \notin I \right\}.$$

The group Γ_μ defined in Lemma 2.3 acts on $\tilde{U}_\mu$. Consider, for a given element $\underline{w} \in \tilde{U}_\mu$, the element $\underline{w}^\mu = (w_1^\mu, \cdots, w_d^\mu) \in \mathbb{C}^d$ defined as follows

$$\begin{cases} w_h^\mu = 0 & \text{if} \quad h \in I \\ w_h^\mu = \sqrt{\sum_{j=1}^n a_{jh}\left(|w_{r_j}|^2 + \lambda_{r_j}\right) - \lambda_h} & \text{if} \quad h \notin I. \end{cases}$$

Notice that $\underline{w} + \underline{w}^\mu$ belongs to $\Psi^{-1}(0)$: define $\nu \in \mathfrak{d}^*$ so that $< \nu, X_j >= |w_j|^2 + \lambda_j$, for all $j \in I$; then it is easy to check that $J(\underline{w} + \underline{w}^\mu) = \pi^*(\nu)$ and $\nu \in \Delta$. Consider now the open sets $\hat{U}_\mu = \hat{V}_\mu \cap \Psi^{-1}(0) \subset \Psi^{-1}(0)$ and $U_\mu = \hat{U}_\mu/N \subset \Psi^{-1}(0)/N$. Then the surjective mapping

$$\begin{array}{rccc} q_\mu \colon & \tilde{U}_\mu & \longrightarrow & U_\mu \\ & \underline{w} & \longmapsto & [\underline{w}^\mu + \underline{w}] \end{array}$$

induces a homeomorphism

$$\begin{array}{rccc} \psi_\mu \colon & \tilde{U}_\mu/\Gamma_\mu & \longrightarrow & U_\mu \\ & [\underline{w}] & \longmapsto & [\underline{w}^\mu + \underline{w}]. \end{array}$$

The above data define a chart, and the union of the U_μ's, for μ ranging over all the vertices of Δ, cover the symplectic quotient.

Let us now show that the mapping χ lifts to a diffeomorphism $\tilde{\chi}_\mu \colon \tilde{U}_\mu \to \tilde{V}_\mu$ for each vertex μ. For each $\underline{w} \in \tilde{U}_\mu$, there exists a unique element $a(\underline{w}) \in A$ such that $a(\underline{w}) \cdot (\underline{w} + \underline{w}^\mu)$ is of the form $\underline{z} + \underline{z}^\mu$, where $\underline{z} \in \tilde{V}_\mu$, and $\underline{z}^\mu$ is given by (9). Then

we define $\tilde{\chi}_\mu(\underline{w}) = \underline{z}$. We compute $\underline{z}$ explicitly following Lemma 2.3: $\underline{z} = \tilde{\chi}_\mu(\underline{w}) = \exp\left(-i\pi_\mu^{-1}(\pi_{\mathbb{C}}(C(\underline{w})))\right) \cdot \underline{w}$, where the h-component of $C(w) \in \prod_{h \notin J} \mathbb{C}e_h$ is given by $\frac{1}{4\pi}\log A_h$, with $A_h = \sum_{j=1}^n a_{jh}(|w_{r_j}|^2 + \lambda_{r_j}) - \lambda_h$. Notice that the mapping $\tilde{\chi}_\mu$ thus defined is equivariant with respect to the action of Γ_μ on $\tilde{U}_\mu$ and $\tilde{V}_\mu$, and it is a lift of χ. It is now straightforward to deduce the bijectivity of $\tilde{\chi}_\mu$ from that of χ, using the explicit expression for $\tilde{\chi}_\mu$. In order to prove that $\tilde{\chi}_\mu$ is a diffeomorphism we apply the inverse function theorem. Therefore we only have to check that its Jacobian matrix $D(\tilde{\chi}_\mu)$ is non-degenerate on $\tilde{U}_\mu$. Let $(\underline{x}, \underline{y})$ be real coordinates in $\tilde{U}_\mu$, it turns out that the Jacobian matrix has the following form

$$D(\tilde{\chi}_\mu)(\underline{x}, \underline{y}) = e^r \left(I_{2n} + \begin{pmatrix} M \\ N \end{pmatrix} ({}^tM, {}^tN) \right),$$

where r is the function $\sum_{h \notin I} a_{kh} \frac{1}{2} \log A_h$, and where M, N are $(n, d-n)$ matrices with entries $u_k a_{kh} \frac{1}{\sqrt{A_h}}$ and $v_k a_{kh} \frac{1}{\sqrt{A_h}}$ respectively. This implies that $D(\tilde{\chi}_\mu)(w)$ is positive definite for every $\underline{w} \in \tilde{U}_\mu$. To conclude that χ is a diffeomorphism observe that the continuity of the equivariant mapping $\tilde{\chi}_\mu^{-1}$ for each vertex μ implies that χ^{-1} is continuous, since ϕ_μ and ψ_μ are homeomorphisms.

Now, having proved that χ is a diffeomorphism, we can consider the complex quotient endowed with a symplectic and a complex structure. We want to prove that the symplectic form is Kähler. This can be checked pointwise: let $\hat{\underline{w}} = \underline{w} + \underline{w}^\mu$ be a point in $\widehat{U}_\mu$. Then the N-orbit through $\hat{\underline{w}}$ is contained in $\Psi^{-1}(0)$ and the A-orbit through $\hat{\underline{w}}$ is orthogonal to $\Psi^{-1}(0)$. This gives the isomorphism

$$T_{\hat{\underline{w}}}(\Psi^{-1}(0))/T_{\hat{\underline{w}}}(N \cdot \hat{\underline{w}}) \simeq \mathbb{C}^d/T_{\hat{\underline{w}}}(N_{\mathbb{C}} \cdot \hat{\underline{w}}).$$

To conclude the proof we only have to remark that the symplectic structure on $\Psi^{-1}(0)/N$ and the complex structure on $\mathbb{C}_\Delta^d/N_{\mathbb{C}}$, read on $T_{\hat{\underline{w}}}(\Psi^{-1}(0))/T_{\hat{\underline{w}}}(N \cdot \hat{\underline{w}})$, are exactly the ones induced by the standard complex and symplectic structures of $\mathbb{C}^d$. $\square$

Remark 3.7 Consider the complex quasifold $\mathbb{C}_\Delta^d/N_{\mathbb{C}}$ constructed in Theorem 2.2. Notice that, by varying the coefficients λ_j in (2), we can produce many simple convex polytopes, each allowing the same choice of inward-pointing normals X_j and of quasilattice Q that we had made for Δ (for example we can "inflate" Δ). The corresponding complex quasifolds are therefore exactly the same, but on the symplectic side in general one obtains non-equivalent symplectic structures. In other terms we obtain many non-isometric Kähler structures on the same complex quasifold $\mathbb{C}_\Delta^d/N_{\mathbb{C}}$.

We conclude this section with the discussion of an example. The unit interval provides a very useful model for a thorough understanding of the diffeomorphism χ and a neat example of Kähler quasifold.

Example 3.8 Consider the polytope $[0, 1] \subset \mathbb{R}^*$, with the same choice of vectors and quasilattice made in Example 2.6 (we refer to that example for the notation). We can associate to these data the complex quasifold $\mathbb{C}_\Delta^2/N_{\mathbb{C}}$ and also a symplectic quasifold,

the *quasisphere* of [P, Examples 1.13, 1.19, 2.10 and 3.5], of which we recall only some features.

Consider the symplectic quasifold $\Psi^{-1}(0)/N$, where

$$\Psi^{-1}(0) = \{\, (z,w) \in \mathbb{C}^2 \mid t|z_1|^2 + s|z_2|^2 = st \,\},$$

and

$$N = \{\, (e^{2\pi iX}, e^{2\pi i\frac{s}{t}X}) \mid X \in \mathbb{R} \,\}.$$

The space $\Psi^{-1}(0)/N$ is covered by two charts

$$U_{\mathrm{S}} = \left\{\, [z_1 : z_2] \in \Psi^{-1}(0)/N \mid z_2 \neq 0 \,\right\} \quad \text{and} \quad U_{\mathrm{N}} = \left\{\, [z_1 : z_2] \in \Psi^{-1}(0)/N \mid z_1 \neq 0 \,\right\}.$$

The corresponding local models are defined by $\tilde{U}_{\mathrm{S}} = \{z \in \mathbb{C} \mid |z| < \sqrt{s}\} \subset \tilde{V}_{\mathrm{S}}$ acted on by Γ_{S} and by $\tilde{U}_{\mathrm{N}} = \{z \in \mathbb{C} \mid |z| < \sqrt{t}\} \subset \tilde{V}_{\mathrm{N}}$ acted on by Γ_{N}. To complete the picture we need the homeomorphisms:

$$\begin{array}{cccc} \Phi_{\mathrm{S}}: & \tilde{U}_{\mathrm{S}}/\Gamma_{\mathrm{S}} & \longrightarrow & U_{\mathrm{S}} \\ & [z] & \longmapsto & \left[z : \sqrt{t - \frac{t}{s}|z|^2}\right] \end{array}, \qquad \begin{array}{cccc} \Phi_{\mathrm{N}}: & \tilde{U}_{\mathrm{N}}/\Gamma_{\mathrm{N}} & \longrightarrow & U_{\mathrm{N}} \\ & [w] & \longmapsto & \left[\sqrt{s - \frac{s}{t}|w|^2} : w\right] \end{array}.$$

Now we are ready to write the local lifts $\tilde{\chi}_{\mathrm{S}}$ and $\tilde{\chi}_{\mathrm{N}}$ of the diffeomorphism χ. Notice that

$$(\chi \circ \Phi_{\mathrm{S}})([z]) = \left[z : \sqrt{t - \frac{t}{s}|z|^2}\right]_{N_{\mathbb{C}}} = \left[z(t - \frac{t}{s}|z|^2)^{-\frac{s}{2t}} : 1\right]_{N_{\mathbb{C}}}.$$

Therefore a local lift $\tilde{\chi}_{\mathrm{S}} : \tilde{U}_{\mathrm{S}} \longrightarrow \tilde{V}_{\mathrm{S}}$ is given by the equivariant diffeomorphism

$$\tilde{\chi}_{\mathrm{S}}(z) = z \left(t - \frac{t}{s}|z|^2\right)^{-\frac{s}{2t}}$$

Analogously a local lift $\tilde{\chi}_{\mathrm{N}} : \tilde{U}_{\mathrm{N}} \longrightarrow \tilde{V}_{\mathrm{N}}$ is given by the equivariant diffeomorphism

$$\tilde{\chi}_{\mathrm{N}}(w) = w \left(s - \frac{s}{t}|w|^2\right)^{-\frac{t}{2s}}.$$

We exhibit now the complex quotient as a Kähler quasifold. We define the Kähler form ω by giving its local lifts on $\tilde{V}_{\mathrm{S}}$ and $\tilde{V}_{\mathrm{N}}$:

$$\tilde{\omega}_{\mathrm{S}} = \frac{1}{2\pi i}\frac{s}{t}\frac{1}{\left(\frac{1}{s} + \frac{1}{t}|z|^{(1+\frac{s}{t})}\right)^2} dz \wedge d\bar{z}, \qquad \tilde{\omega}_{\mathrm{N}} = \frac{1}{2\pi i}\frac{t}{s}\frac{1}{\left(\frac{1}{t} + \frac{1}{s}|w|^{(1+\frac{t}{s})}\right)^2} dw \wedge d\overline{w}. \quad (15)$$

They are invariant under the action of Γ_{S} and Γ_{N} respectively and it is a straightforward computation to check that $\tilde{\chi}_{\mathrm{S}}^*\tilde{\omega}_{\mathrm{S}} = \frac{1}{2\pi i} dz \wedge d\bar{z}$ and $\tilde{\chi}_{\mathrm{N}}^*\tilde{\omega}_{\mathrm{N}} = \frac{1}{2\pi i} dw \wedge d\overline{w}$. On the other hand these are precisely the local lifts of the symplectic form on $\tilde{U}_{\mathrm{S}}$ and $\tilde{U}_{\mathrm{N}}$ respectively, so ω is the pullback via χ^{-1} of the symplectic form on $\Psi^{-1}(0)/N$ and hence defines a Kähler form on the complex quotient.

To conclude the discussion of this significant example, it is worthwhile to check directly that the local forms given in (15) fulfill the definition of differential form. We

have to show that they behave correctly under the change of charts. The first move is to pull them back to $W_{\mathrm{S}}^{\#}$ and to $W_{\mathrm{N}}^{\#}$, which are both equal to $\mathbb{C}$. We obtain

$$\omega_{\mathrm{S}}^{\#} = \frac{1}{2\pi i} \frac{s}{t} \frac{e^{(\zeta+\overline{\zeta})}}{\left(\frac{1}{s} + \frac{1}{t} e^{(\zeta+\overline{\zeta})\frac{1}{2}(1+\frac{s}{t})}\right)^2} \, d\zeta \wedge d\overline{\zeta}, \qquad \omega_{\mathrm{N}}^{\#} = \frac{1}{2\pi i} \frac{t}{s} \frac{1}{\left(\frac{1}{t} + \frac{1}{s} e^{(\eta+\overline{\eta})\frac{1}{2}(1+\frac{t}{s})}\right)^2} \, d\eta \wedge d\overline{\eta}$$

which are invariant under the respective action of Λ_{S}, Λ_{N}. It is easy to check that the mapping $\zeta \mapsto \eta = -\frac{s}{t}\zeta$ pulls $\omega_{\mathrm{N}}^{\#}$ back to $\omega_{\mathrm{S}}^{\#}$.

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DIPARTIMENTO DI MATEMATICA APPLICATA "G. SANSONE", VIA S. MARTA 3, 50139 FIRENZE, ITALY, `mailto:fiamma@dma.unifi.it`
AND
LABORATOIRE DIEUDONNÉ, UNIVERSITÉ DE NICE, PARC VALROSE, 06108 NICE CEDEX 2, FRANCE, `mailto:elisa@alum.mit.edu`